Bethe-Heitler cross-section for very high photon energies and large muon scattering angles

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Abstract. The cross-section for the process $\gamma + A \to \mu^+ + \mu^- + X$ is studied where the photon energy is of the order of several hundreds of GeV and where one of the leptons is scattered to large angles. This is of practical importance for muon shielding calculations at future linear electron colliders. In addition to the photon pole contribution which was previously considered especially by Y.S.Tsai, we identify another component due to the muon pole (equivalent photon and equivalent muon approximation). This is discussed following the usual Bethe-Heitler formula. Then we give practically useful formulae for inclusive lepton (muon) production along with some numerical examples.

1 Introduction

This paper arose out of a practical question. At future Linear Colliders like TESLA [1] electron and positron beams of several hundreds of GeV and high beam powers of some 10 MW have to be absorbed after having passed the interaction region. In this process, high energy photons are produced which in turn give rise to high energy muons. In order to estimate the muon radiation dose at earth surface above the Linear Collider [2] pair production cross-sections for large angles are necessary. This is a well known problem which has been studied many times, most elaborately probably by Y.S.Tsai [4]. An exact lowest order formula is given there, which corresponds to the graphs shown in Figs. 1 a) and b). The information about nuclear structure is fully contained in the electromagnetic structure functions W_1 and W_2. However, this formula is very complicated and hard to evaluate practically, especially if one has to integrate over the unobserved lepton with fourmomentum p_+ (e or μ, we are here concerned only with muons). Therefore practical calculations were done using the Weizsäcker-Williams (or equivalent photon) approximation. This corresponds to the kinematical situation where the square of the momentum of the exchanged photon, $q^2 = -Q^2$, is specially small ($q^2 \approx 0$). In many situations, this is the dominant distribution. In our studies we found another kinematical situation to be important. It corresponds to the muon pole, where the intermediate muon is close to its mass-shell ("equivalent muon approximation"). This has been studied before [6] (see also [8]) and we adopt the formulations of these authors.
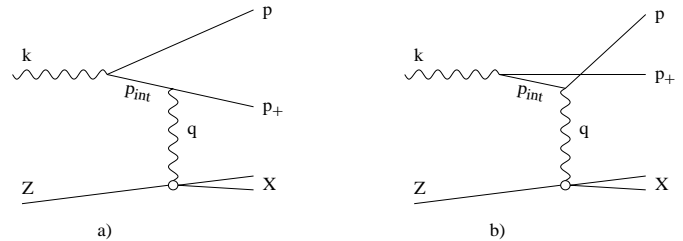

Fig. 1. Feynman graphs describing the process
$\gamma + A \to p + p_+ + X$

Starting from the usual formula for the Bethe-Heitler process (where infinitely heavy point-like nuclei are assumed) we explain in chapter 2 the general features of the pair production process and its important limiting cases. Then we discuss the general case, where structure effects are taken into account (along with the effects due to the finite mass, or recoil effects). The information needed as an input is sufficiently well known, specially from electron scattering. In chapter 3 we provide some illustrative examples, together with discussions. Our conclusions are given in chapter 4.

2 Cross-section for $\gamma + A \to \mu^+ + \mu^- + X$

2.1 Bethe-Heitler cross-section revisited. Photon and muon pole contributions

The cross-section for the Bethe-Heitler process is calculated in many textbooks corresponding to the graphs of Fig. 1. Assuming that the nucleus is infinitely heavy and

point-like one obtains the following formula for the differential cross-section (we use the natural units $\hbar = c = 1$) [3]

$$d\sigma = \frac{8\alpha^3 Z^2 m^2}{\pi k^3 Q^4} E_+ \, E dE \tag{1}$$

$$\left\{ -\frac{\delta_+^2}{(1+\delta_+^2)^2} - \frac{\delta^2}{(1+\delta^2)^2} \right.$$

$$+ \frac{k^2}{2E_+ E} \frac{\delta_+^2 + \delta^2}{(1+\delta_+^2)(1+\delta^2)}$$

$$\left. + \left(\frac{E_+}{E} + \frac{E}{E_+} \right) \frac{\delta_+ \delta \cos(\phi)}{(1+\delta_+^2)(1+\delta^2)} \right\} \delta_+ d\delta_+ \, \delta d\delta \, d\phi$$

where m denotes the muon mass, $k = E_+ + E$ is the photon energy, E_+, E the energies of the outgoing muons and ϕ is the angle between the planes spanned by the photon and the outgoing muons. We have $\delta_+ = \theta_+ E_+/m$, $\delta = \theta E/m$, where θ_+, θ are the scattering angles of the outgoing muons, and the momentum transfer is given by

$$Q^2/m^2 = \delta_+^2 + \delta^2 + 2\delta_+ \delta \cos(\phi)$$

$$+ m^2 \left(\frac{1+\delta_+^2}{2E_+} + \frac{1+\delta^2}{2E} \right)^2 \tag{2}$$

Typical scattering angles are given by $\theta \cong m/E$, i.e. $\delta \cong 1$. Now we are interested in the case where one of the muons (say μ^-) scatters to large angles and we integrate over the angles of the other one (θ_+, ϕ), i.e. we have

$$\delta \gg 1 \tag{3}$$

In this integration, a large contribution will come from the region where Q^2 is as small as possible. This is the case for

$$\delta_+ \cong \delta \qquad \text{and} \qquad \phi \cong \pi \tag{4}$$

i.e. μ^+ and μ^- scatter to opposite sides with about equal transverse momentum. In this case (2) leads to

$$Q^2_{min} \cong m^4 (1+\delta^2)^2 \left(\frac{k}{2E_+ E} \right)^2 \tag{5}$$

and one uses the Weizsäcker-Williams approximation as it is elaborated in [4].

In addition, there is another region of integration which can become important,

$$\delta \gg 1 \quad , \quad 0 \le \delta_+ \le \delta_{+,max} \quad \text{and} \quad 0 \le \phi < 2\pi \tag{6}$$

where the choice of $\delta_{+,max}$ is discussed below. In this case the momentum transfer is given by

$$Q^2_{mp} \cong m^2 \delta^2 = (\theta E)^2 \tag{7}$$

which is generally much larger than Q^2_{min}. The paranthesis in equ.(1) is simplified and one obtains for the differential cross-section (integrated over the angles of the μ^+, $u \equiv \delta_+^2$, $u_{max} = \delta_{+,max}^2$)

$$\frac{d^2\sigma}{d\Omega dE} = \frac{4Z^2}{\pi} \frac{\alpha^3 E_+ E^3}{k^3 Q^4_{mp}}$$

$$\int_0^{u_{max}} du \left(\frac{-u}{(1+u)^2} + \frac{k^2}{2E_+ E(1+u)} \right) \tag{8}$$

The integral in equ.(8) diverges logarithmically. Following [6] (see also [8]) we put $\delta_{+,max} = \theta E/m \gg 1$ and we find

$$\frac{d^2\sigma}{d\Omega dE} \cong \left[\frac{4Z^2\alpha^2}{\theta^4 E^2} \right] \frac{\alpha}{\pi} \ln \left(\theta \frac{E}{m} \right) \frac{E^2 + (k-E)^2}{k^3} \tag{9}$$

where $E_+ = k - E$ was used. This is in agreement with [8] (see especially equ.(13)) and with [6] (see (6)-(8)). The derivation of equ.(9) actually depends on the assumption that $\delta_+ \ll \delta$. In the next subsection we give a simple physical meaning to equ.(9) in a more general context.

2.2 Effects of finite nuclear size, nuclear and nucleon structure

For large scattering angles, $\delta \gg 1$, the photon and muon pole contributions are separated in phase space, since $\delta_+ \cong \delta \gg 1$ for the equivalent photon approximation (EPA) and $\delta_+ \le 1$ for the equivalent muon approximation (EMA). Therefore we can add these two terms incoherently to obtain the full inclusive cross-section

$$\frac{d^2\sigma}{d\Omega dE} = \left(\frac{d^2\sigma}{d\Omega dE} \right)_{EPA} + \left(\frac{d^2\sigma}{d\Omega dE} \right)_{EMA} \tag{10}$$

The relative importance of the two terms depends on the special kinematical values and the Q^2-dependence of the corresponding structure functions.

The photon pole contribution was extensivly discussed in [4] and we follow this procedure and do not have to go into details here. There are coherent nuclear scattering, described by the elastic form-factor of the nucleus and, for larger values of Q^2_{min} (i.e. larger scattering angles), incoherent contributions due to the scattering from individual nucleons[1]. In the equivalent photon approximation, the effects of nucleon structure are entirely contained in the function χ, see eqn.(2.19) of [5]

$$\chi = \frac{1}{2M_i} \int_{M_i^2}^{\infty} dM_f^2 \int_{Q'^2_{min}}^{Q^2_{max}} \frac{dQ^2}{Q^4} \left[(Q^2 - Q'^2_{min}) \cdot W_2(Q^2, M_f^2) \right.$$

$$\left. + 2Q^2_{min} \cdot W_1(Q^2, M_f^2) \right] \tag{11}$$

The mass of the hadron system in the initial state is denoted by M_i, the mass of the produced final state by M_f, and

$$Q'^2_{min} = Q^2_{min} + 2\Delta\sqrt{Q^2_{min}} \quad \text{where} \quad \Delta = \frac{M_f^2 - M_i^2}{2M_i}. \tag{12}$$

The expression for Q^2_{min} (see (5)) is in accord with the corresponding expression given in App.A of [5]. The electromagnetic structure functions are denoted by W_2 and W_1. The elastic contribution is well described by the usual dipole parametrization (see [4]). In this reference, a parametrization of the inelastic contribution $M_f \ne M_i$ is also

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[1]   There is also a "quasi-elastic" contribution where the Pauli-suppression effect on the knocked out nucleon is included.

given ("meson production form-factor"). We suggest a somewhat different approach, which takes the resonant character of the structure function for $Q^2 \approx 0$ into account. For small enough Q^2 (the important region of the integration in (11)) the structure functions are directly related to the cross-section for real (transverse) photons $\sigma_{\gamma p}(k)$ (the scalar part vanishes as $Q^2 \to 0$). One has

$$W_2(Q^2, M_f^2) \cong \frac{Q^2}{4\pi^2 \alpha} \frac{2M_i}{M_f^2 - M_i^2} \sigma_{\gamma p}(M_f) f(Q^2) \quad (13)$$

$$W_1(Q^2, M_f^2) \cong \frac{1}{4\pi^2 \alpha} \frac{M_f^2 - M_i^2}{2M_i} \sigma_{\gamma p}(M_f) f(Q^2)$$

where a form-factor $f(Q^2)$ is introduced. The cross-section $\sigma_{\gamma p}$ for real photons is dominated by nucleon resonances, most prominently by the Δ-resonance. We take the Q^2-dependence of $f(Q^2)$ to be the same as for the elastic proton form-factor, i.e. we choose a dipole form

$$f(Q^2) = \frac{1}{(1 + Q^2/\Lambda^2)^4} \quad \text{where} \quad \Lambda^2 = 0.71\,\text{GeV}^2 . \quad (14)$$

Note, however, that a somewhat stronger fall-off with Q^2 was found experimentally [9].

By applying the equivalent photon approximation to the full Bethe-Heitler expression, one neglects the muon pole contribution. In many cases this is justified because the Q^2-value involved in the muon pole contribution is usual much higher compared to the one occuring in the equivalent photon approximation. Form-factor effects emphasizing the low Q^2-values tend to make the muon pole contribution small. On the other hand, in the deep-inelastic scattering region scaling sets in and the structure functions do not decrease any more with increasing Q^2.

The muon pole contribution is described in [6,8] (equivalent muon approximation). After integration over the unobserved μ^+ (which is scattered to small angles), the scattering process factorizes into an "equivalent muon spectrum" and the scattering cross-section of the muon on the target. This muon can be considered as a "parton" inside the photon. It is moving in the direction of the photon with an energy fraction $x = E'/k$ of the photon, where E' corresponds to the muon energy in the intermediate state. The inclusive Bethe-Heitler cross-section is now obtained as (see (25) of [6])

$$\left(\frac{d^2\sigma}{d\Omega dE}\right)_{EMA} = \int_{x_o}^1 dx\, F_\gamma^\mu(k, x)\, \frac{d^2\sigma(\mu N \to \mu' X)}{d\Omega dE}(kx) \quad (15)$$

This equivalent muon spectrum is given by [6]

$$F_\gamma^\mu(k, x) = \frac{\alpha}{\pi}\, \ln\left(\frac{k}{m}\right) \left[x^2 + (1-x)^2\right] \quad (16)$$

A more refined expression is obtained by replacing

$$\ln\left(\frac{k}{m}\right) \quad \text{by} \quad \ln\left(\theta \frac{xk}{m}\right) \quad \text{with} \quad \theta \gg \frac{m}{xk} \quad (17)$$

(see also [6] and [8]). The muon-nucleon inclusive scattering cross-section is given by

$$\frac{d^2\sigma(\mu N \to \mu' X)}{d\Omega dE} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \cdot \left(W_2 + 2W_1 \tan^2(\theta/2)\right) \quad (18)$$

where the Mott cross-section is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \frac{\alpha^2}{4(kx)^2} \frac{\cos^2(\theta/2)}{\sin^4(\theta/2)} \quad (19)$$

where the muon energy is given by $E' = xk$. The kinematical limit x_o in (16) is given by [6]

$$x_o = \frac{M_N E}{M_N k - 2kE \sin^2(\theta/2)} \quad (20)$$

in the present notation. Equ.(15) is rewritten as

$$\left(\frac{d^2\sigma}{d\Omega dE}\right)_{EMA} = \frac{\alpha^2}{4k^2 \sin^4(\theta/2)} \cdot \quad (21)$$
$$[W_2^\gamma(k, E, \theta) \cos^2(\theta/2) + 2W_1^\gamma(k, E, \theta) \sin^2(\theta/2)]$$

with

$$W_{1,2}^\gamma(k, E, \theta) = \frac{\alpha}{\pi} \int_{x_o}^1 dx\, \ln\left(\theta \frac{xk}{m}\right) \frac{x^2 + (1-x)^2}{x^2}$$
$$\times W_{1,2}(\nu, Q^2) \quad (22)$$

where $\nu = xk - E$ and $Q^2 = 4xkE \sin^2(\theta/2)$ are the usual variables used in deep inelastic scattering. In order to evaluate the integral in (22) we have to know the structure function on the ray

$$Q^2 = 4(\nu + E)E \sin^2(\theta/2) \quad (23)$$

For an infinitely heavy and point-like target proton we have $W_2 = \delta(xk - E)$. The integration (21),(22) leads to the same cross-section as it was discussed in Sect. 2.1, (9) (for $Z = 1$), if one applies the small angle approximation ($\theta \ll 1$). Now one sees (9) factorizes into the Mott scattering cross-section (with $E = E' = xk$) and the equivalent muon number of (16).

In principle, these structure functions are well known from deep inelasic lepton scattering [11],[10]. For a rough estimate of the order of magnitude of the effect we use a simplified approach: for the rather low values of $Q^2 \geq 1\text{GeV}^2$ and $M_f \geq 2.6\text{GeV}$ an approximate scaling behaviour sets in [11]. We have

$$\nu W_2(\nu, Q^2) = F_2(x_q) \quad (24)$$

where $x_q = Q^2/(2\nu M_N)$ and F_2 is independent of Q^2. We also have

$$2M_N W_1(\nu, Q^2) = F_1(x_q) \quad (25)$$

and the Callan-Gross relation

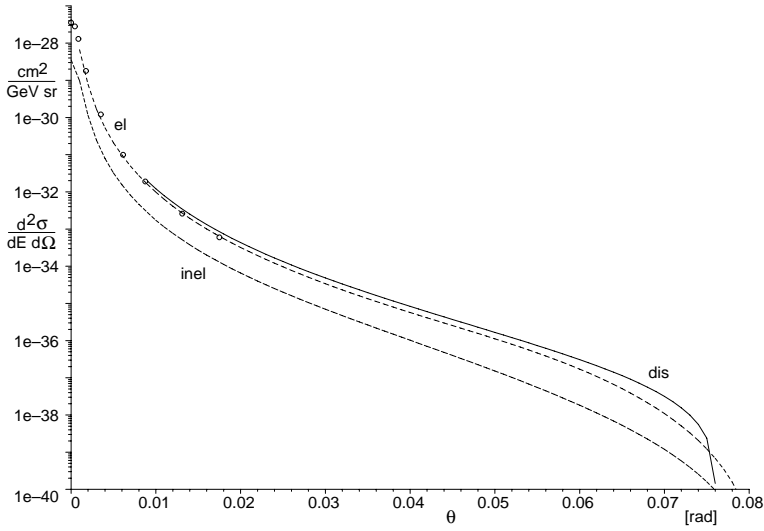$$F_2(x_q) = 2x_q F_1(x_q). \quad (26)$$

Fig. 2. Bethe Heitler cross-sections on a proton at a photon energy of 200 GeV and a muon energy of 120 GeV as a function of the polar angle θ. Solid line – deep inelastic (dis), dashed lines – elastic (el) and inelastic (inel). For further explanations see text

Table 1. Numerically calculated integrals of (27)

M_1 [GeV]	M_2 [GeV]	M_f [GeV]	$\tilde{W}_2$ [GeV^{-2}]	$\tilde{W}_1$
1.11	1.35	1.24	2.66	0.30
1.35	1.62	1.49	1.24	0.62
1.62	2.05	1.82	1.00	1.67
2.05	2.55	2.31	0.65	3.56
2.55	3.25	2.81	0.62	10.09

3 Numerical results and discussion

For small scattering angles, the Bethe-Heitler cross-section is dominated by coherent nuclear scattering. With increasing angles, the effect of the form-factor of the nucleus will set in and the cross-section will decrease rather rapidly. The nuclear form-factor is characterized by a rather soft scale $\Lambda^2_{nucleus} = \hbar c/R = 0.005 \, \text{GeV}^2$ for R=3 fm. The corresponding scale for a nucleon is $\Lambda^2_N = 0.71 \, \text{GeV}^2$. For large angles, the incoherent scattering from the nucleons will take over. Apart from Pauli-blocking effects, the nucleus is just an assembly of Z protons and N neutrons at rest (we can neglect Fermi motion). This is all very well and extensively described in [4] and we can concentrate on the incoherent contributions from the nucleons.

The elastic nucleon contribution is well described within the equivalent photon approximation [4]. We take the usual dipole form-factors. We have also checked the muon pole contribution for the elastic case. Since the momentum transfer Q^2_{min} (see (5)) is much smaller than Q^2_{mp} (see (7)) the strong decrease of the dipole form-factor with Q^2 renders the muon pole contribution negligible in this case.

Inelastic contributions for small Q^2 are dominated by nucleon resonances. We use (11) and (13) to calculate the inelastic contribution. We take the cross-section $\sigma_{\gamma p}$ for real photons from experimental data [10]. This cross-section is dominated by nucleon resonances (mainly the

Δ) in the GeV region, followed by a structureless continuum. For the integration over M the cross-section $\sigma_{\gamma p}$ is regarded as a sequence of 5 resonances at the center of mass energies M_f as listed in tab.1. The first 2 are the real resonances. The relative widths of all the regions are kept approximately the same. The following integrals corresponding to equ.(13) were numerically calculated for each region:

$$\tilde{W}_2 = \frac{1}{4\pi^2\alpha} \int_{M_1}^{M_2} 2M \, dM \, \frac{1}{M^2 - M_i^2} \, \sigma_{\gamma p}(M) \qquad (27)$$

$$\tilde{W}_1 = \frac{1}{4\pi^2\alpha} \frac{1}{(2M_i)^2} \int_{M_1}^{M_2} 2M \, dM \, (M^2 - M_i^2) \, \sigma_{\gamma p}(M)$$

$$M_f = \int_{M_1}^{M_2} dM \, M \, \sigma_{\gamma p}(M) \bigg/ \int_{M_1}^{M_2} dM \, \sigma_{\gamma p}(M)$$

They are listed in tab.1. A parametrization of the contribution of the Δ-resonance to W_2 is given by Chanfray et al. [7], our results are in qualitative agreement. At large angles the resulting cross-section behaves very similarly to the elastic cross-section (shown in Fig. 2, dashed lines) as expected from the Q^2-dependencies of the form factors. It should be noted here that proton and neutron behave similarly as it is indicated by comparing the γp with the γd cross-section [10].

We concentrate now on the muon pole contribution. Due to the quark-parton structure of the nucleon, the form factor does not decrease any more for high Q^2 values. This is the reason why the muon pole contribution is important. For (roughly) $Q^2 > 1 \, \text{GeV}^2$ and $M_f > 2.6 \, \text{GeV}$ scaling sets in [11]. Following [12] we put

$$F_2(x_q) = \rho \ln(1/x_q) \qquad \text{with} \qquad \rho = 0.16 \qquad (28)$$

This is a very rough approximation which has the merit that it leads to an analytical expression. For our present

purposes this seems sufficient. Using (21)-(26) and (28) we get for the deep inelastic (dis) cross-section

$$\left(\frac{d^2\sigma}{d\Omega dE}\right)_{dis} = \frac{\alpha^3}{4\pi k^2 \sin^4(\theta/2)} \cdot \rho \cdot \int_{x_o}^{1} dx \ \ln\left(\theta\frac{xk}{m}\right)$$

$$\cdot \frac{x^2 + (1-x)^2}{x^2} \cdot \ln\frac{2M_N(xk-E)}{xkE\theta^2}$$

$$\cdot \left(\frac{1}{xk-E} + \frac{xk-E}{xkE}\right) \quad (29)$$

At the expense of using more computer time, more sophisticated expressions for the structure function $F_2(x_q)$ can be inserted. The structure functions of the neutron are also known [13],[14]. A simple approximation is given in [15] $F_{2,neutron} = (1-x_q)F_{2,proton}$, valid for $x_q < 0.75$. For $x_q > 0.75$ is $F_{2,neutron} \cong F_{2,proton}$ (see Fig. 8 of [13]). Since $F_{2,proton}$ is always larger than $F_{2,neutron}$, one would only overestimate the cross section, when treating all nucleons as protons. In the following we restrict ourselves to the Bethe-Heitler process on a proton. As a typical case we take $k = 200\,\text{GeV}$. The energy of the outgoing muon is, following Tsai [4], taken to be 120 GeV. For smaller angles,we can also compare our calculations to Tsai. We give 3 contributions

The elastic production. Here the photon pole contribution dominates over the muon pole contribution. For the latter the momentum transfer is considerably larger and form-factor effects render it negligible (this was checked, but we do not need to show it here). Fig. 2 showes our elastic cross-section in good agreement to Tsai's values [4] (small circles).

The inelastic contribution. ("meson production form factor"). We calculated the cross-sections of each of the 5 bins listed in tab.1 and found the contributions from W_1 negligible compared to W_2. The sum is shown in Fig. 2. The Δ-resonance (1st bin) is the dominant contribution with some 50% at small and more than 80% at large angles. This is expected, since the strength $\tilde{W}_2$ is largest and M_f is lowest. The contributions of the other bins decrease with increasing M_f.

The deep inelastic contribution. For small angles, Q^2 becomes less than $Q_0^2 = 1\,\text{GeV}^2$ and our simple parametrizations of $W_{1,2}$ break down. This is approximately the case for $\theta < \theta_{min} = Q_0/E$ (independent of k). Therefore we start our calculation at θ_{min} as it is shown in Fig. 2 (solid line). For these small angles, the other contributions are already dominant. Furthermore, the condition $M_f > 2.6\,\text{GeV}$ has to be fulfilled, when scaling is applicable. This means that the lower limit x_0 in (20) is shifted to a somewhat larger value

$$x_o' = \frac{1/2(M_{f,min}^2 - M_N^2) + M_N E}{M_N k - 2Ek\sin^2(\theta/2)} \quad (30)$$

In general, this is a small (less than 10%) effect, since the muon energy region where E is not much larger than the nucleon mass M_N is already excluded by applying θ_{min}.

As it can be seen in Fig. 2 the deep inelastic contribution is very important for large muon scattering angles.

4 Conclusions

In conclusion we have presented a new practical approach to the Bethe Heitler process for large scattering angles at high energies. Of course, in an exact evaluation of the Bethe Heitler expression, as it is given in [4] (equ.(2.3)) the muon pole contribution is included. However this expression is too cumbersome and time consuming for practical purposes. We have shown how to include the "deep inelastic contribution" and show that it is important for relevant numerical examples. This extends the results of Tsai [4]. Among other things, such contributions are important for muon shielding problems at future linear colliders.

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